Quantum master equations from classical Lagrangians with two stochastic forces

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We show how a large family of master equations, describing quantum Brownian motion of a harmonic oscillator with translationally invariant damping, can be derived within a phenomenological approach, based on the assumption that an environment can be simulated by two classical stochastic forces. This family is determined by three time-dependent correlation functions (besides the frequency and damping coefficients), and it includes as special cases the known master equations, whose dissipative part is bilinear with respect to the operators of coordinate and momentum.

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I. INTRODUCTION

The phenomena of irreversibility and damping in quantum systems have been the subjects of numerous studies from the very first years of Quantum Mechanics (QM) [1, 2, 3, 4]. They are attributed to the action of some environment that drains information from quantum system $\mathcal{S}$, thus increasing the entropy during its evolution. The environment is usually referred as reservoir $\mathcal{R}$ whose variables are expressed as q-numbers (operators acting in the Hilbert space), and it is assumed to have an infinity of degrees of freedom. In the most general case, one has to treat the whole quantum system $\mathcal{S}+\mathcal{R}$, taking into account all details of each subsystem and the interaction between them. It is known, however, that under certain conditions the influence of environment can be described approximately by means of a few parameters which enter some dynamical equations containing only the variables related to the system $\mathcal{S}$. In the Schrödinger picture such equations for the statistical operator $\hat{\rho}(t)$ or its different representations (density matrix, Wigner function, etc.), known under the name *quantum master equations* (QME), were studied in numerous papers, e.g., [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. An alternative description in the Heisenberg picture is achieved within the frameworks of *quantum Langevin equations* or Langevin-Heisenberg equations (LHE's) [16, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39]. These two approaches were discussed in detail in many books and reviews [40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52].

The shortest and simplest *formal* way of introducing relaxation in quantum mechanics is to postulate some general structure of the master or Langevin–Heisenberg equations, which guarantees fulfilment of the fundamental requirements of QM (such as the properties of hermiticity and positivity of the statistical operator, as well as its normalization; or preservation of canonical commutation relations between the time-dependent operators) for any instant of time. For example, a general structure of dynamical mappings of density operators preserving positive semidefiniteness was established in [53, 54], and equivalent differential equations were considered in [55, 56, 57, 58]. For a more restricted problem of *quantum Brownian motion* this approach was used, e.g., in Refs. [59, 60, 61, 62, 63, 64], where some sets of free parameters were chosen in such a way that *mean values* of coordinates and momenta obeyed given classical equations of motion.

However, it seems desirable to have some schemes permitting to *derive* master equations from some general principles. The most natural way to do this is to start from some explicit Hamiltonian for the total system $\mathcal{S}+\mathcal{R}$. Then the dynamical equations for the subsystem $\mathcal{S}$ arise as a result of taking the trace over the reservoir variables. In this approach both the subsystem and reservoir are considered as quantum objects from the very beginning. A frequently used simple model of the reservoir consists of an infinite number of harmonic oscillators (HO's) with a frequency distribution chosen according to some hypothesis. Note, however, that although the use of a reservoir is usually viewed as a "microscopic" approach for the derivation of LHE or QME, in reality, unless one knows *precisely* the nature of the environment and its interaction with $\mathcal{S}$ (described by a Hamiltonian with parameters dependent on fundamental constants; see, e.g., [65] for this exceptional case), strictly speaking, it should be considered as *phenomenological*. This occurs because several assumptions must be made concerning the very nature of the reservoir modes, on the parameters of the reservoir Hamiltonian and on the $\mathcal{R}-\mathcal{S}$ interaction term.

Taking this point of view, it seems interesting to study, what kinds of master equations can be obtained, if one follows the phenomenological path from the very beginning to the very end. This is the aim of our paper. As a general principle we assume the Lagrange–Hamiltonian formalism, which impelled the development of physics in general and in particular QM and field theories. It is

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complemented by the assumption that $\mathcal{S}$ interacts with time-dependent (TD) classical fluctuating forces $[F_i(t)]$, which simulate the action of the environment without any further preoccupation about the nature of their microscopic origin. We show that combining some phenomenological Lagrangian, which takes into account the effect of dissipation, with a suitably chosen set of classical stochastic forces and following standard rules of the "canonical quantization" procedure, one can obtain in a quite straightforward and simple manner a large class of master equations, which embraces many equations, introduced earlier within the frameworks of different schemes. This approach was used earlier in Refs.[66, 67, 68, 69, 70]. However, in those papers the authors considered only *one* classical stochastic force. As a result, the equations obtained had some restrictions on their domain of validity. We show that introducing *two* stochastic forces one can obtain more general equations.

The paper is organized as follows. In Sec. II we give a brief review of known master equations describing quantum damped oscillators. New results are contained in Sec. III, where we derive the master equation with the use of two effective classical fluctuating forces. We show that in the classical case, it is always possible to use a single force, without changing the physics. However, in the quantum case two forces are necessary to obtain equations compatible with the principles of QM. Sec. IV contains a summary and conclusions. In appendix A we expose some details of calculations.

II. CONVENTIONAL MASTER EQUATIONS

The Hamiltonian H_0 considered in this paper corresponds to a particle of mass m and angular frequency ω_0 subjected to a one-dimensional harmonic force,

$$\hat{H}_0 = \hbar\omega_0 \left(\hat{a}^\dagger \hat{a} + 1/2\right) = \hat{p}^2/(2m) + m\omega_0^2 \hat{x}^2/2, \qquad (1)$$

where the non-hermitian lowering operator $\hat{a}$ can be written in terms of hermitian operators, the particle position $\hat{x}$ and its momentum $\hat{p}$, as

$$\hat{a} = (2\hbar\omega_0 m)^{-1/2} \left(m\omega_0 \hat{x} + i\hat{p}\right).$$

The general form of the master equation is

$$d\hat{\rho}/dt + (i/\hbar)\,[H, \hat{\rho}] = \mathcal{L}\hat{\rho}, \qquad (2)$$

where the dissipative nonunitary superoperator $\mathcal{L}$ accounts for the influence of the environment and the operator H can be different from H_0. One of the most frequently used master equations for the damped harmonic oscillator has an origin in the problems of quantum optics. It corresponds to $H = H_0$ and

$$\begin{aligned}\mathcal{L}\hat{\rho} &= \lambda(\bar{n}+1)\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right) \\ &\quad +\lambda\bar{n}\left(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\right), \end{aligned} \qquad (3)$$

where $\bar{n} \equiv \bar{n}(\omega_0, T) = (e^\beta - 1)^{-1}$ is the reservoir mean number of quanta at temperature T ($\beta = \hbar\omega_0/k_B T$), k_B is Boltzmann constant and λ is the decay rate or damping constant, which contains the strength parameter of the interaction between $\mathcal{R}$ and $\mathcal{S}$. The operator (3) seems to be derived for the first time under the assumption of *weak coupling* between a quantum system $\mathcal{S}$ and a quantum *thermal* reservoir and by adopting the Born-Markov approximation in [15, 16, 17, 18]. Its representation in the Fock basis was derived in the general form in [8, 9, 10, 19], although the special case of the equation for *diagonal matrix elements* can be traced to papers by Landau, Bloch and Pauli [1, 2, 3] (see also [6]).

An immediate generalization of operator (3) is

$$\mathcal{L}\hat{\rho} = \sum_j \left(2\hat{\Phi}_j\hat{\rho}\hat{\Phi}_j^\dagger - \hat{\Phi}_j^\dagger\hat{\Phi}_j\hat{\rho} - \hat{\rho}\hat{\Phi}_j^\dagger\hat{\Phi}_j\right), \qquad (4)$$

where $\hat{\Phi}_j$ are arbitrary linear operators (their number also may be arbitrary). Sometimes the right-hand side of (4) is called the "Lindblad form", after the study [58], although this general structure, preserving the hermiticity, normalization and positivity of $\hat{\rho}$, was discovered by several authors earlier [55, 56, 57].

The relaxation operator in terms of operators $\hat{x}$ and $\hat{p}$ is usually associated with the problem of *quantum Brownian motion*. The most general master equation, preserving the normalization and hermiticity of the statistical operator $\hat{\rho}$ and containing only *bilinear* forms of operators $\hat{x}$ and $\hat{p}$, corresponds to the choice

$$H = H_0 + \frac{\mu}{2}\{\hat{x}, \hat{p}\}, \qquad (5)$$

$$\begin{aligned}\mathcal{L}\hat{\rho} &= \frac{i\lambda}{2\hbar}[\hat{p}, \{\hat{x}, \hat{\rho}\}] - \frac{i\lambda}{2\hbar}[\hat{x}, \{\hat{p}, \hat{\rho}\}] - \frac{D_p}{\hbar^2}[\hat{x}, [\hat{x}, \hat{\rho}]] \\ &\quad -\frac{D_x}{\hbar^2}[\hat{p}, [\hat{p}, \hat{\rho}]] + \frac{D_z}{\hbar^2}[\hat{x}, [\hat{p}, \hat{\rho}]] + \frac{D_z}{\hbar^2}[\hat{p}, [\hat{x}, \hat{\rho}]] \end{aligned} \qquad (6)$$

where μ, λ, D_x, D_p and D_z can be, in principle, arbitrary functions of time. This form was considered, e.g., in Refs. [30, 46, 61] (in the case of time-independent coefficients) and [37] (with arbitrary time-dependent coefficients). Some differences in explicit expressions can be removed with the aid of identities $[\hat{x}, \hat{p}] = i\hbar$ and

$$[\hat{x}, \{\hat{p}, \hat{\rho}\}] + [\hat{p}, \{\hat{x}, \hat{p}\}] + [\hat{p}, \{\hat{\rho}, \hat{x}\}] = 0.$$

The meaning of parameters λ and μ becomes clear from the equations for mean values of the coordinate and momentum:

$$d\langle\hat{x}\rangle/dt = \langle\hat{p}\rangle/m + (\mu - \lambda)\langle\hat{x}\rangle, \qquad (7)$$

$$d\langle\hat{p}\rangle/dt = -m\omega_0^2\langle\hat{x}\rangle - (\mu + \lambda)\langle\hat{p}\rangle. \qquad (8)$$

The choice $\lambda = \mu$ eliminates the friction term from the equation for $d\langle\hat{x}\rangle/dt$. This special case was studied by Dekker in [68]. Generalizations of (5) and (6) to multidimensional systems (in particular, to the case of a charged

particle in a magnetic field) were given in [59, 60]. The superoperator (3) is a particular case of (6) for

$$D_p = \frac{\lambda}{2} m\hbar\omega_0 \left(\bar{n} + \frac{1}{2}\right) = (m\omega_0)^2 D_x, \quad D_z = \mu = 0.$$

The operator master equation (6), being written in terms of the Wigner function,

$$W(x,p,t) = \frac{1}{\pi\hbar} \int dy e^{-2ipy/\hbar} \langle x - y|\hat{\rho}(t)|x + y\rangle, \quad (9)$$

assumes a simple form of the Fokker–Planck equation,

$$\begin{aligned} \frac{\partial W}{\partial t} &= \frac{\partial}{\partial p}\left(\left[m\omega_0^2 x + (\mu+\lambda)p\right] W\right) \\ &\quad -\frac{\partial}{\partial x}\left(\left[p/m + (\mu-\lambda)x\right] W\right) \\ &\quad + D_p \frac{\partial^2 W}{\partial p^2} + D_x \frac{\partial^2 W}{\partial x^2} + 2D_z \frac{\partial^2 W}{\partial p \partial x}. \quad (10) \end{aligned}$$

Thus we see that the terms proportional to D_x, D_p, and D_z in Eq. (6) describe the diffusion in the phase space.

Introducing the phase space vector variable $\mathbf{q} = (x, p)$, one can write Eq. (10) in a compact form

$$\frac{\partial W(\mathbf{q},t)}{\partial t} = -\frac{\partial}{\partial q_i}\left[(\mathbf{A}\mathbf{q})_i W\right] + D_{ij}\frac{\partial^2 W}{\partial q_i \partial q_j} \quad (11)$$

(sum over repeated indices is understood), where

$$\mathbf{A} = \left\| \begin{array}{cc} \mu - \lambda & m^{-1} \\ -m\omega_0^2 & -(\mu+\lambda) \end{array} \right\| \quad (12)$$

is a 'drift' matrix, which governs the evolution of the first order statistical moments (mean values),

$$d\langle\mathbf{q}\rangle/dt = \mathbf{A}\langle\mathbf{q}\rangle. \quad (13)$$

Introducing the covariances $\sigma_{jk} = \frac{1}{2}\langle\hat{q}_j\hat{q}_k + \hat{q}_k\hat{q}_j\rangle - \langle\hat{q}_j\rangle\langle\hat{q}_k\rangle$, one can verify that both equations, (10) and (11), result in the following equation for the symmetrical covariance matrix $\mathbf{M} \equiv \|\sigma_{jk}\|$:

$$d\mathbf{M}/dt = \mathbf{A}\mathbf{M} + \mathbf{M}\tilde{\mathbf{A}} + 2\mathbf{D}, \quad (14)$$

where $\tilde{\mathbf{A}}$ is the transposed matrix of $\mathbf{A}$ and $\mathbf{D} \equiv \|D_{ij}\|$ is the symmetrical diffusion matrix ($D_{12} = D_{21} = D_z$).

Taking the operators $\hat{\Phi}_j$ in Eq. (4) as linear combinations of operators $\hat{x}$ and $\hat{p}$,

$$\hat{\Phi}_j = \alpha_j\hat{x} + \beta_j\hat{p}, \quad (15)$$

one can verify that the operator (6) can be rewritten in the form (4), provided the following conditions can be satisfied [71]:

$$\sum_j |\beta_j|^2 = D_x/\hbar^2, \quad \sum_j |\alpha_j|^2 = D_p/\hbar^2, \quad (16)$$

$$\sum_j \alpha_j^*\beta_j = i\lambda/(2\hbar) - D_z/\hbar^2. \quad (17)$$

In view of the Schwartz inequality,

$$\sum_j |\beta_j|^2 \sum_j |\alpha_j|^2 \geq |\sum_j \alpha_j^*\beta_j|^2,$$

the condition of compatibility of Eqs. (16) and (17) is the inequality

$$D_p D_x - D_z^2 \geq (\hbar\lambda/2)^2 \equiv [\hbar\mathrm{Tr}(\mathbf{A})/4]^2. \quad (18)$$

The condition (18), which was derived and discussed from different points of view in Refs. [52, 59, 60, 61, 62, 63, 71, 72, 73], guarantees that the positivity of statistical operator is preserved for all times and *for any physically admissible initial state*. This is the necessary and sufficient condition (together with conditions $D_x \geq 0$ and $D_p \geq 0$) of reducibility of the operator (6) to the Lindblad form (4) [59, 60]. Note that parameter μ does not enter the constraint (18), because it is related to the correction to the Hamiltonian (5) and not to the non-unitary part of the total Liouville superoperator. So its presence in the "friction forces" in Eqs. (7) and (8) is not relevant to the existence or non-existence of the Lindblad representation of the master equation.

Some frequently considered master equations with *time-independent coefficients*, such as, e. g., the Agarwal equation [74] (with $\lambda = \mu$, $D_p = 2m\lambda\omega_0\bar{n}$ and $D_x = D_z = 0$) or its special case, known as the Caldeira–Leggett equation [75] (with $\lambda = \mu$, $D_p = 2m\lambda k_B T$ and $D_x = D_z = 0$), do not satisfy the condition (18). Consequently, these equations can result in violations of the positivity of the statistical operator (which is equivalent to the violation of the uncertainty relations [47, 60, 70, 71]) at the intermediate stages of evolution, if one tries to use them outside the domain of their validity (which corresponds to the limit of high temperatures, $k_B T \gg \hbar\omega_0$).

On the other hand, it was shown in [76, 77, 78, 79, 80] that the Wigner function of a subsystem $\mathcal{S}$ interacting with a reservoir $\mathcal{R}$ satisfies Eq. (11) *at any time*, if, (I) the total Hamiltonian of the whole system $\mathcal{S} + \mathcal{R}$ is an arbitrary quadratic form with respect to coordinates and momenta (in particular, the interaction Hamiltonian can be an arbitrary bilinear form with respect to the coordinates of $\mathcal{S}$ and $\mathcal{R}$) and if, (II) the initial statistical operator of the total system is factorized, $\hat{\rho}_{tot} = \hat{\rho}_{\mathcal{S}}\hat{\rho}_{\mathcal{R}}$, where $\hat{\rho}_{\mathcal{R}}$ is an arbitrary *Gaussian* state (i.e., not necessarily thermal, it can be *squeezed*, for example). However, in such a case, (a) the matrices $\mathbf{A}$ and $\mathbf{D}$ are *explicitly time-dependent* (which is interpreted sometimes as a manifestation of *non-Markovian evolution* [37, 81, 82]), and (b) the constraint (18) (or its multidimensional generalizations [59, 60]) can be violated. This does not mean that the state $\hat{\rho}_{\mathcal{S}}(t)$ can become unphysical – no, simply in this case the system $\mathcal{S}$ does not pass over *all possible* mixed states in the process of evolution, but it moves only along

some specific trajectories in the Hilbert space; see in this connection also [36, 76, 83].

Time-independent diffusion and drift matrices appear only *asymptotically*, as $t \to \infty$ (physically, after some characteristic time determined by the properties of $\mathcal{R}$). Moreover, the sets of diffusion coefficients satisfying (18) can be obtained *only for specific forms of the* $\mathcal{R}-\mathcal{S}$ *interaction Hamiltonian*. For example, in the case of a thermal reservoir, the relaxation superoperator (2) can be derived if the interaction only has the so called *rotating wave approximation* form, which results in the drift matrix (12) with $\mu = 0$ [79]. The superoperator (6), with *arbitrary* time independent diffusion coefficients, can be derived with the use of *squeezed* (or *rigged*) reservoirs [33, 84]. Our main goal in this paper is to find the sets of drift and diffusion coefficients (possibly, time-dependent) that can be obtained from the scheme of quantization of classical equations with *two* stochastic forces.

III. FROM CLASSICAL STOCHASTIC FORCES TO QUANTUM MASTER EQUATIONS

A. Classical treatment

A typical equation of motion of a particle of mass m subjected to a linear friction force in one dimension is

$$\ddot{x} + \dot{\Gamma}_t \dot{x} + \frac{1}{m}\frac{\partial V(x,t)}{\partial x} = 0, \tag{19}$$

where $V(x,t)$ is a potential and Γ_t a TD dissipative function. Although Eq. (19) describes a nonconservative system, it is known for a long time (see, e.g., [85, 86] and references therein) that it can be derived from some Lagrangian. The most simple one is known under the name Bateman–Caldirola–Kanai Lagrangian [87, 88],

$$L(x,\dot{x},t) = \left(\frac{1}{2}m\dot{x}^2 - V(x,t)\right)e^{\Gamma_t}. \tag{20}$$

We consider a simple generalization of (20),

$$L(x,\dot{x},t) = \left(\frac{1}{2}m\dot{x}^2 - V(x,t) + xF_t + \dot{x}G_t\right)e^{\Gamma_t}, \tag{21}$$

where F_t and G_t are arbitrary generalized TD forces associated to the position x and to the velocity $\dot{x}$. The *canonical* momentum is

$$P \equiv \partial L/\partial \dot{x} = e^{\Gamma_t}(m\dot{x} + G_t) \tag{22}$$

and we define the *physical* momentum as

$$p \equiv m\dot{x} + G_t = Pe^{-\Gamma_t}. \tag{23}$$

The Hamiltonian associated to the Lagrangian (21) is

$$\begin{aligned} H(t) &= P\dot{x} - L = \frac{P^2}{2m}e^{-\Gamma_t} + [V(x,t) - xF_t]\,e^{\Gamma_t} \\ &\quad -PG_t/m + G_t^2\exp(\Gamma_t)/2m. \end{aligned} \tag{24}$$

The classical equations of motion for canonical coordinates (Hamilton equations) are

$$\begin{aligned} \dot{x} &= \partial H(t)/\partial P = \left(Pe^{-\Gamma_t} - G_t\right)/m, \\ \dot{P} &= -\partial H(t)/\partial x = \left(-\partial V(x,t)/\partial x + F_t\right)e^{\Gamma_t}, \end{aligned}$$

whereas for the physical coordinates one obtains

$$\dot{x} = p/m - G_t/m, \qquad \dot{p} = -\partial V(x,t)/\partial x + F_t - p\dot{\Gamma}_t,$$

or as a single second order equation (Newton equation)

$$\ddot{x} + \dot{\Gamma}_t\dot{x} + \frac{1}{m}\frac{\partial V(x,t)}{\partial x} = \frac{1}{m}\left(F_t - \dot{\Gamma}_t G_t - \dot{G}_t\right), \tag{25}$$

where the RHS contains TD terms only.

We see that in classical mechanics, where the coordinate x is the only independent variable (since p or P are functions of $\dot{x}$), the presence of two terms, $-xF_te^{\Gamma_t}$ and $-PG_t/m$, in Hamiltonian (24) is redundant, because the dynamics depends only on the combination $\mathcal{F}(t) = F_t - \dot{\Gamma}_tG_t - \dot{G}_t$. A usual choice is $G_t \equiv 0$ and $F_t = \mathcal{F}(t)$. But one can obtain the same dynamics, choosing $F_t \equiv 0$ and finding the function G_t from the equation $-(\dot{\Gamma}_tG_t + \dot{G}_t) = \mathcal{F}(t)$, whose solution is

$$G_t \equiv Ke^{-\Gamma_t} - e^{-\Gamma_t}\int^t e^{\Gamma_\tau}\mathcal{F}(\tau)d\tau,$$

where K is an arbitrary constant.

However, both the forces, F_t and G_t, are important in the quantum case, because of the non-commutation property of position and momentum. These forces give different contributions to the dynamical evolution of the system state, as we show in the following subsection.

B. Quantum treatment

Having the Hamiltonian function (24), one can try to "quantize" the classical dissipative system, transforming (24) to an operator by means of the usual rules and writing the time-dependent Schrödinger equation with this "quantum" Hamilton operator. This idea was formulated for the first time by Caldirola and Kanai (CK) in 1940s [88], and since that time it was developed or criticized by many authors (see, e.g., Refs. [46, 89]. It is known by now that such a simplified approach suffers from many drawbacks. For instance, the CK Hamiltonian is explicitly time-dependent, so it is closer to the system with time-dependent mass than to the genuine dissipative system. Moreover, the problem of finding the Hamiltonian for the given equations of motion has no unique solution, and practically all such Hamiltonians have some pathology [85, 86]. But the main physical defect of the CK scheme is that it implies that the quantum state of the system remains *pure* during the evolution, because regular classical fields interacting with a quantum system $\mathcal{S}$ do not change its informational content as time goes

on, even if energy is not conserved; an initial pure state $\psi_{\mathcal{S}}(0)$ will evolves as a pure state $\psi_{\mathcal{S}}(t)$. On the other hand, it is known that dissipation is connected with a loss of quantum purity. Thus one has to describe the system in terms of the density matrix or its equivalent forms, such as the Wigner function, for example. But how to find equations of motion for the density matrix?

An answer was given in Refs. [66, 67, 68, 69, 70]: one should start not from the Schrödinger equation for a wave function, but from the von Neumann–Liouville equation for the statistical operator, considering F_t and G_t as *stochastic forces* and performing some averaging over these forces. This averaging results in an information drain from the quantum system $\mathcal{S}$. So irreversibility is verified and the entropy of the system changes due to the random character of the classical fields. However, by using a *single* stochastic force one obtains equations which not always preserve the property of positivity of the statistical operator. Our goal here is to show that correct equations, satisfying all principles of QM, can be derived in a very simple way, if one introduces *two* classical stochastic forces, instead of a single one, besides one regular dissipative function. Such a derivation was not done earlier, as far as we know.

The equation for the time evolution of the density matrix following from Hamiltonian (24) reads

$$\frac{d\hat{\rho}_t}{dt} = \frac{1}{i\hbar}\left[\hat{H}_{ef}(t) + \hat{W}(\hat{x},\hat{P},t), \hat{\rho}_t\right], \tag{26}$$

where

$$\hat{H}_{ef}(t) = e^{-\Gamma_t}\hat{P}^2/(2m) + V(\hat{x},t)\, e^{\Gamma_t}, \tag{27}$$

$$\hat{W}(\hat{x},\hat{P},t) = -e^{\Gamma_t}\hat{x}F_t - \hat{P}G_t/m. \tag{28}$$

$\hat{W}(\hat{x},\hat{P},t)$ is a stochastic operator if F_t and G_t are assumed as stochastic forces. The TD term $G_t^2\exp(\Gamma_t)/2m$ was thrown out from $\hat{H}_{ef}(t)$ since it does not contribute to the equations of motion.

Using the unitary evolution operator $\hat{U}_t$ corresponding to the effective free Hamiltonian (27),

$$i\hbar\, d\hat{U}_t/dt = \hat{H}_{ef}(t)\,\hat{U}_t,$$

we make a unitary transformation

$$\hat{\rho}_t = \hat{U}_t\tilde{\rho}_t\hat{U}_t^{\dagger}, \tag{29}$$

which removes the term $\hat{H}_{ef}(t)$ from Eq. (26):

$$i\hbar\, d\tilde{\rho}_t/dt = \left[\hat{W}(\tilde{x}_t,\tilde{P}_t,t), \tilde{\rho}_t\right] \equiv \left[\tilde{W}(t), \tilde{\rho}_t\right], \tag{30}$$

and where

$$\tilde{x}_t = U_t^{\dagger}\hat{x}U_t, \qquad \tilde{P}_t = U_t^{\dagger}\hat{P}U_t. \tag{31}$$

A formal solution to Eq. (30) is

$$\tilde{\rho}_t = \hat{\rho}_0 + \frac{1}{i\hbar}\int_0^t dt'\left[\tilde{W}(t'), \tilde{\rho}_{t'}\right]. \tag{32}$$

Iterating Eq. (32) and deriving with respect to time we get the equation

$$\begin{aligned}\frac{d\tilde{\rho}_t}{dt} &= \frac{1}{i\hbar}\left[\tilde{W}(t), \hat{\rho}_0\right] \\ &\quad + \frac{1}{(i\hbar)^2}\int_0^t dt'\left[\tilde{W}(t), \left[\tilde{W}(t'), \tilde{\rho}_{t'}\right]\right]\end{aligned} \tag{33}$$

which is still exact. Its physical content is the same as Eq. (30). Using the RHS of (32) for $\tilde{\rho}_{t'}$ and inserting it in (33) recursively, we obtain an infinite series

$$\begin{aligned}\frac{d\tilde{\rho}_t}{dt} &= \frac{1}{i\hbar}\left[\tilde{W}(t), \hat{\rho}_0\right] + \sum_{k=1}^{\infty}\frac{1}{(i\hbar)^{k+1}}\int_0^t dt_1\cdots\int_0^{t_{k-1}} dt_k \\ &\quad \times\left[\tilde{W}(t), \left[\tilde{W}(t_1), \cdots\left[\tilde{W}(t_k), \hat{\rho}_0\right]\right]\right].\end{aligned} \tag{34}$$

The next step is to perform averaging over stochastic forces in Eq. (34). We assume that classical stochastic functions F_t and G_t are Gaussian with zero average values, $\overline{F_t} = \overline{G_t} = 0$, and that they are delta-correlated with a TD function (not a stationary process):

$$\overline{F_{t_1}F_{t_2}} = 2A_{t_1}\delta(t_1 - t_2), \tag{35}$$

$$\overline{G_{t_1}G_{t_2}} = 2m^2B_{t_1}\delta(t_1 - t_2), \tag{36}$$

$$\overline{F_{t_1}G_{t_2}} = 2mC_{t_1}\delta(t_1 - t_2). \tag{37}$$

Moreover, we take into account the important properties of *Gaussian* stochastic processes, namely, $\overline{J_{t_1}J_{t_2}...J_{t_{2n+1}}} = 0$ for an odd number of terms (J_t stands for F_t or G_t), whereas for an even number

$$\overline{J_{t_1}J_{t_2}...J_{t_{2n}}} = \sum_{all\ pairs}\overline{J_{t_i}J_{t_j}}\cdot\overline{J_{t_k}J_{t_l}}, \tag{38}$$

where the average is over ensembles. Since stochastic operators $\tilde{W}(t_k)$ are linear combinations of $\tilde{x}_{t_k}$ and $\tilde{P}_{t_k}$, only the terms with even numbers of operators $\tilde{W}(t_k)$ survive after the averaging in Eq. (34), so that we arrive at the series containing only even powers of $\hbar$:

$$\frac{d\hat{\rho}_t}{dt} = \sum_{k=1}^{\infty}\frac{1}{(i\hbar)^{2k}}\hat{\chi}_{2k}(t). \tag{39}$$

The first term of this expansion is a sum of four integrals containing double commutators,

$$\begin{aligned}\hat{\chi}_2(t) &= \int_0^t dt' e^{\Gamma_t+\Gamma_{t'}}\overline{F_tF_{t'}}\,[\tilde{x}_t, [\tilde{x}_{t'}, \hat{\rho}_0]] \\ &\quad + m^{-1}e^{\Gamma_t}\int_0^t dt'\overline{F_tG_{t'}}\left[\tilde{x}_t, \left[\tilde{P}_{t'}, \hat{\rho}_0\right]\right] \\ &\quad + m^{-1}\int_0^t dt' e^{\Gamma_{t'}}\overline{G_tF_{t'}}\left[\tilde{P}_t, [\tilde{x}_{t'}, \hat{\rho}_0]\right] \\ &\quad + m^{-2}\int_0^t dt'\overline{G_tG_{t'}}\left[\tilde{P}_t, \left[\tilde{P}_{t'}, \hat{\rho}_0\right]\right],\end{aligned}$$

which can be easily calculated due to the presence of delta-functions in Eqs. (35)–(37), resulting in the following expression:

$$\hat{\chi}_2(t) = A_t e^{2\Gamma_t}[\tilde{x}_t,[\tilde{x}_t,\hat{\rho}_0]] + B_t\left[\tilde{P}_t,\left[\tilde{P}_t,\hat{\rho}_0\right]\right] + C_t e^{\Gamma_t}\left(\left[\tilde{x}_t,\left[\tilde{P}_t,\hat{\rho}_0\right]\right] + \left[\tilde{P}_t,[\tilde{x}_t,\hat{\rho}_0]\right]\right) \quad (40)$$

Continuing these steps, we see that the structure of the term $\hat{\chi}_2(t)$ is repeated each time, resulting finally in replacing the initial operator $\hat{\rho}_0$ by the time dependent operator $\hat{\rho}_t$. Thus we obtain the following closed equation governing the time evolution of the statistical operator averaged over stochastic forces (see Appendix A for details of the derivation):

$$\frac{d\hat{\rho}_t}{dt} = \frac{1}{i\hbar}\left[\hat{H}_{ef}(t),\hat{\rho}_t\right] - \frac{1}{\hbar^2}\left(A_t e^{2\Gamma_t}[\hat{x},[\hat{x},\hat{\rho}_t]] + B_t[\hat{P},[\hat{P},\hat{\rho}_t]] + 2C_t e^{\Gamma_t}[\hat{x},[\hat{P},\hat{\rho}_t]]\right). \quad (41)$$

We would like to emphasize that no truncations of higher order terms were done in deriving Eq. (41), so this equation holds for any TD force strengths (coefficients A_t, B_t, C_t). A possible additional term proportional to $[\hat{P},[\hat{x},\hat{\rho}_t]]$ in the RHS of Eq. (41) is redundant because of the identity $[\hat{x},[\hat{P},\hat{\rho}_t]] = [\hat{P},[\hat{x},\hat{\rho}_t]]$. Had we assumed one force F_t only, the coefficients B_t and C_t would be zero. Equation (41) is structurally analogous to that obtained by Hu, Paz and Zhang [78], although their TD coefficients were derived assuming a reservoir made of HO's, while ours are purely phenomenological. Below we show that the associated Wigner functions coincide, and as such Eq. (41) contains the non-Markovian effects (in the sense of Refs. [76, 78, 81, 82], i.e., time-dependent diffusion and drift coefficients). In our derivation, limits as high or low-temperature, strong or weak-coupling have no room for discussion since all the effects of the environment on the system depend on the adopted values for the four TD parameters. In particular, for coefficients A, B and C being time-independent one retrieves the Markovian limit.

Eq. (41) cannot be immediately identified with Eq. (6), due to different meanings of the variables $\hat{p}$ (physical momentum) in (6) and $\hat{P}$ (canonical momentum) in (41), besides the presence of the factors $\exp(\Gamma_t)$ and $\exp(2\Gamma_t)$. However, it is easy to show that these equations are physically equivalent, because they give the same Wigner function for the mapped physical coordinates.

The equations of motion of first and second moments for the canonical variables are (the average values are defined as $\langle\hat{A}\rangle \equiv \mathrm{Tr}(\hat{A}\hat{\rho}_t)$)

$$d\langle\hat{x}\rangle/dt = e^{-\Gamma_t}\langle\hat{P}\rangle/m, \qquad d\langle\hat{P}\rangle/dt = \langle -\partial V/\partial x\rangle e^{\Gamma_t},$$

$$d\langle\hat{x}^2\rangle/dt = e^{-\Gamma_t}\langle\{\hat{x},\hat{P}\}\rangle/m + 2B_t,$$

$$d\langle\hat{P}^2\rangle/dt = -\langle\{\hat{P},\partial\hat{V}/\partial x\}\rangle e^{\Gamma_t} + 2A_t e^{2\Gamma_t},$$

$$\frac{d}{dt}\langle\{\hat{x},\hat{P}\}\rangle = \frac{2}{m}\langle\hat{P}^2\rangle e^{-\Gamma_t} - 2\langle\hat{x}\frac{\partial\hat{V}}{\partial x}\rangle e^{\Gamma_t} + 4C_t e^{\Gamma_t},$$

and one notices that these equations contain the TD exponential factors $\exp(\pm\Gamma_t)$. However, passing to the physical momentum (23), we get rid of these factors,

$$d\langle\hat{x}\rangle/dt = \langle\hat{p}\rangle/m, \quad (42)$$

$$d\langle\hat{p}\rangle/dt = -\langle\partial\hat{V}/\partial\hat{x}\rangle - \dot{\Gamma}_t\langle\hat{p}\rangle, \quad (43)$$

$$d\langle\hat{x}^2\rangle/dt = m^{-1}\langle\{\hat{x},\hat{p}\}\rangle + 2B_t, \quad (44)$$

$$d\langle\hat{p}^2\rangle/dt = -\langle\{\hat{p},\partial\hat{V}/\partial\hat{x}\}\rangle - 2\dot{\Gamma}_t\langle\hat{p}^2\rangle + 2A_t, \quad (45)$$

$$\frac{d}{dt}\langle\{\hat{x},\hat{p}\}\rangle = \frac{2}{m}\langle\hat{p}^2\rangle - 2\langle x\frac{\partial\hat{V}}{\partial\hat{x}}\rangle - \dot{\Gamma}_t\langle\{\hat{x},\hat{p}\}\rangle + 4C_t. \quad (46)$$

Comparing Eqs. (12) and (13) with (42) and (43), we see that they coincide if $\hat{V}(x) = m\omega_0^2\hat{x}^2/2$, $\Gamma_t = 2\lambda t$, and $\mu = \lambda$. Then comparing equations for the covariances of the physical momentum and coordinate, one can verify that they satisfy the matrix equation (14) if the diffusion coefficients are identified as

$$A_t = D_x(t), \quad B_t = D_p(t), \quad C_t = D_z(t). \quad (47)$$

In what follows we will show the equivalence between the Wigner function $W^P(\mathbf{Q},t)$ in the *canonical* phase space, $\mathbf{Q} = (x,P)$ with the Wigner function in the physical variables phase space $W(\mathbf{q},t)$. The $W^P(\mathbf{Q},t)$ is governed by the Fokker–Planck equation (11) with time-dependent drift and diffusion matrices,

$$\mathbf{A}^P = \begin{Vmatrix} 0 & e^{-\lambda t}/m \\ -m\omega_0^2 e^{\lambda t} & 0 \end{Vmatrix}, \quad \mathbf{D}^P = \begin{Vmatrix} B_t & C_t e^{\lambda t} \\ C_t e^{\lambda t} & A_t e^{2\lambda t} \end{Vmatrix}.$$

The solution of Eq. (11) for the function $W(\mathbf{Q},t)$ can be written as

$$W^P(\mathbf{Q},t) = \int G^P(\mathbf{Q},\mathbf{Q}',t)W^P(\mathbf{Q}',0)d\mathbf{Q}',$$

where the propagator is given by the formula [40, 52]

$$G^P(\mathbf{Q},\mathbf{Q}',t) = \left(2\pi\sqrt{\det\mathbf{N}(t)}\right)^{-1} \times \exp\left[-\frac{1}{2}(\mathbf{Q}-\mathbf{R}^P(t)\mathbf{Q}')\mathbf{N}^{-1}(t)(\mathbf{Q}-\mathbf{R}^P(t)\mathbf{Q}')\right]. \quad (48)$$

Matrix $\mathbf{N}(t)$ satisfies Eq. (14) (with matrices $\mathbf{A}^P$ and $\mathbf{D}^P$) and the initial condition $\mathbf{N}(0) = 0$, whereas matrix $\mathbf{R}^P(t)$ satisfies the equation $d\mathbf{R}^P/dt = \mathbf{A}^P(t)\mathbf{R}^P$ and the initial condition $\mathbf{R}^P(0) = \mathbf{1}$ (unity matrix).

Differential equations for three different matrix elements of the symmetrical matrix $\mathbf{N}(t)$ have the form

$$\begin{aligned} dN_{11}/dt &= 2e^{-\lambda t}N_{12}/m + 2B_t, \\ dN_{12}/dt &= e^{-\lambda t}N_{22}/m - m\omega_0^2 e^{\lambda t}N_{11} + 2C_t e^{\lambda t}, \\ dN_{22}/dt &= -2m\omega_0^2 e^{\lambda t}N_{12} + 2A_t e^{2\lambda t}. \end{aligned}$$

Doing the substitution (47) and the changes

$$N_{22} = e^{2\lambda t}M_{22}, \quad N_{12} = e^{\lambda t}M_{12}, \quad N_{11} = M_{11}$$

we obtain the equations for elements of matrix $\mathbf{M}(t)$ with *time-independent* drift and diffusion matrices, given by Eqs. (12) and (14) with $\lambda = \mu$. Then one can verify that the propagators in the canonical and physical phase spaces are related by a simple formula

$$G^P(\mathbf{Q}, \mathbf{Q}', t) = e^{-\lambda t}G(\mathbf{q}, \mathbf{q}', t). \tag{49}$$

Therefore, as the initial Wigner function is the same in both coordinate systems, $W^P(\mathbf{Q}, 0) = W(\mathbf{q}, 0)$, the Wigner function at any time becomes

$$\begin{aligned} W^P(\mathbf{Q}, t) &= \int G^P(\mathbf{Q}; \mathbf{Q}', t)W^P(\mathbf{Q}', 0)\, d\,\mathbf{Q}' \\ &= \int e^{-\lambda t}G(x, p; x', p', t)W(x', p'; 0)e^{\lambda t}dx'dp' \\ &= \int G(x, p; x', p', t)W(x', p'; 0)\, dx'dp' \\ &= W(\mathbf{q}, t). \end{aligned} \tag{50}$$

Thus the master equation (41) is completely equivalent to the translationally invariant ($\lambda = \mu$) form of the master equation for quantum Brownian motion (2) with operators (5) and (6).

IV. SUMMARY AND CONCLUSIONS

We have analyzed the phenomenological approach to build a master equation for describing the irreversible and dissipative dynamical evolution of the state of a quantum system $\mathcal{S}$, under the influence of an unspecified environment. In contradistinction to the microscopic approach that models the environment as a reservoir $\mathcal{R}$ made of an infinite number of degrees of freedom (for example, harmonic oscillators), the phenomenological approach makes use of dissipative functions and stochastic forces. We showed that the Newton equation of motion for $\mathcal{S}$ does not change by introducing *two* such forces instead of only one, however when we do the quantization of the system, both forces become quite important, contributing on equal footing. We derived from first principles, with a Lagrangian containing *one* dissipative function and *two* stochastic forces, the master equation describing the quantum Brownian motion (of a harmonic oscillator) with translationally invariant damping and the most general bilinear (with respect to the coordinate and momentum operators) relaxation superoperator, which can be reduced to an equivalent differential equation of the Fokker–Planck type. However, the TD phenomenological parameters entering the forces cannot be determined within the framework of the phenomenological approach; they should be fixed either from experimental data that reproduce relevant physical properties of $\mathcal{S}$ or from some other considerations, such as the requirement of satisfying the positivity constraint (18) or by fitting properties at thermal equilibrium. In this direction we verified that the master equations derived in [76, 77, 78], containing non-Markovian effects (present in their TD coefficients) are accounted in our derivation where the environment is simulated by two effective forces and the dissipative function, instead of assuming an interaction with an infinite set of HO's.

We would like to emphasize that by averaging over the stochastic forces, we did not disregard any term (see Appendix for details). In this sense, the phenomenological derivation of the master equation is as exact as other approaches [78]. Of course, this happened due to the choice of correlation functions in the form of delta functions, albeit multiplied by time-dependent strength factors. The pertinent question, what could happen in the most general non-Markovian case, when correlation functions, such as $A(t_1, t_2)$, are arbitrary functions of the time difference $t_1 - t_2$ (colored noise), requires a separate study. Certainly, the phenomenological approach used in this paper has limitations, because it is based on some effective Lagrangian. Therefore, although it works well for one-dimensional systems (or isotropical multidimensional ones), it will fail for generic multidimensional systems with several independent damping coefficients, because no effective Lagrangian can be found for such systems [85, 86]. This explains also why only a subfamily of master equations (6), restricted by the condition $\lambda = \mu$ (translationally invariant damping), can be obtained within the framework of the scheme used in this paper: there are no effective Lagrangians for $\lambda \neq \mu$ [i.e., for *two* "friction forces" in the classical equations of motion (7) and (8)].

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APPENDIX A: DERIVATION OF THE PHENOMENOLOGICAL MASTER EQUATION

Iterating Eq. (32) once more we get the formal solution

$$\begin{aligned} \tilde{\rho}_t &= \hat{\rho}_0 + \frac{1}{i\hbar}\int_0^t dt_1 \left[\tilde{W}(t_1), \hat{\rho}_0\right] \\ &+ \left(\frac{1}{i\hbar}\right)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \left[\tilde{W}(t_1), \left[\tilde{W}(t_2), \hat{\rho}_{t_2}\right]\right], \end{aligned} \tag{A1}$$

and deriving it with respect to time we get Eq. (33). Using the RHS of (A1) for $\tilde{\rho}_{t'}$, inserting it in the RHS of Eq. (33) and keeping even terms in $\tilde{W}$ (because only the even terms will survive after averaging over the ensemble), Eq. (33) may be written as an infinite series

$$\begin{aligned}\frac{d\tilde{\rho}_t}{dt} &= \frac{1}{(i\hbar)^2}\int_0^t dt'\Big[\tilde{W}(t),\Big[\tilde{W}(t'),\hat{\rho}_0+\frac{1}{(i\hbar)^2}\\ &\times\int_0^{t'}dt_1\int_0^{t_1}dt_1'\Big[\tilde{W}(t_1),\Big[\tilde{W}(t_1'),\hat{\rho}_0\Big]\Big]+\cdots\Big]\Big]\end{aligned}$$

or

$$\begin{aligned}\frac{d\tilde{\rho}_t}{dt} &= \frac{1}{(i\hbar)^2}\int_0^t dt'\Big[\tilde{W}(t),\Big[\tilde{W}(t'),\hat{\rho}_0\Big]\Big]\\ &+\frac{1}{(i\hbar)^2}\int_0^t dt'\Bigg[\tilde{W}(t),\Bigg[\tilde{W}(t'),\sum_{k=1}^{\infty}\frac{1}{(i\hbar)^{2k}}\\ &\times\int_0^{t'}dt_1\int_0^{t_1}dt_1'\ldots\int_0^{t_{k-1}'}dt_k\int_0^{t_k}dt_k'\\ &\times\Big[\tilde{W}(t_1),\Big[\tilde{W}(t_1'),\ldots\Big[\tilde{W}(t_k),\Big[\tilde{W}(t_k'),\hat{\rho}_0\Big]\Big]\Big]\Big]\Big]\Big]\Bigg],\end{aligned}\tag{A2}$$

with $t_0'\equiv t'$. Averaging the quadratic term in $\tilde{W}(\cdot)$ of (A2), we get the four terms of Eq. (40). Averaging the quartic terms in $\tilde{W}(\cdot)$,

$$\begin{aligned}&\frac{1}{(i\hbar)^4}\int_0^t dt'\int_0^{t'}dt_1\int_0^{t_1}dt_1'\\ &\times\overline{\Big[\tilde{W}(t),\Big[\tilde{W}(t'),\Big[\tilde{W}(t_1),\Big[\tilde{W}(t_1'),\hat{\rho}_0\Big]\Big]\Big]\Big]},\end{aligned}$$

we obtain sixteen terms, which together with the four terms of (40) give the expression

$$\begin{aligned}&\frac{1}{(i\hbar)^2}\left\{A_te^{2\Gamma_t}\left[\tilde{x}(t),\left[\tilde{x}(t),\tilde{\rho}_{t'}\right]\right]\right.\\ &+C_tte^{\Gamma_t}\left(\Big[\tilde{x}(t),\Big[\tilde{P}(t),\tilde{\rho}_{t'}\Big]\Big]+\Big[\tilde{P}(t),\left[\tilde{x}(t),\tilde{\rho}_{t'}\right]\Big]\right)\\ &\left.+B_t\Big[\tilde{P}(t),\Big[\tilde{P}(t),\tilde{\rho}_{t'}\Big]\Big]\right\},\end{aligned}$$

where

$$\begin{aligned}\tilde{\rho}_{t'} &= \hat{\rho}_0+\frac{1}{(i\hbar)^2}\int_0^{t'}dt''\ \left\{A_{t''}e^{2\Gamma_{t''}}\left[\tilde{x}_{t''},\left[\tilde{x}_{t''},\hat{\rho}_0\right]\right]\right.\\ &+C_{t''}e^{\Gamma_{t''}}\left(\Big[\tilde{x}_{t''},\Big[\tilde{P}_{t''},\hat{\rho}_0\Big]\Big]+\Big[\tilde{P}_{t''},\left[\tilde{x}_{t''},\hat{\rho}_0\right]\Big]\right)\\ &\left.+B_{t''}\Big[\tilde{P}_{t''},\Big[\tilde{P}_{t''},\hat{\rho}_0\Big]\Big]\right\}+\cdots.\end{aligned}\tag{A3}$$

Finally, after averaging and collecting all terms in Eq. (A2) one obtains the master equation in the interaction picture

$$\begin{aligned}\frac{d\tilde{\rho}_t}{dt} &= \frac{1}{(i\hbar)^2}\Big\{A_te^{2\Gamma_t}\left[\tilde{x}_t,\left[\tilde{x}_t,\tilde{\rho}_t\right]\right]+B_t\Big[\tilde{P}_t,\Big[\tilde{P}_t,\tilde{\rho}_t\Big]\Big]\\ &+C_te^{\Gamma_t}\left(\Big[\tilde{x}_t,\Big[\tilde{P}_t,\tilde{\rho}_t\Big]\Big]+\Big[\tilde{P}_t,\left[\tilde{x}_t,\tilde{\rho}_t\right]\Big]\right)\Big\},\end{aligned}\tag{A4}$$

which is exact. Note that on the RHS of Eq. (A3) the density operator is time-independent whereas it is TD in Eq. (A4).

To illustrate the calculations, let us consider an example of quartic terms of the form

$$\begin{aligned}&\frac{1}{(i\hbar)^4}\int_0^t dt'\ \int_0^{t'}dt''\int_0^{t''}dt'''\exp\left(\Gamma_t+\Gamma_{t'}+\Gamma_{t''}+\Gamma_{t'''}\right)\\ &\times\overline{F_tF_{t'}F_{t''}F_{t'''}}\left[\tilde{x}_t,\left[\tilde{x}_{t'},\left[\tilde{x}_{t''},\left[\tilde{x}_{t'''},\hat{\rho}_0\right]\right]\right]\right].\end{aligned}$$

They give the following terms in the master equation:

$$\begin{aligned}\frac{d\tilde{\rho}_t}{dt} &= \frac{2}{(i\hbar)^2}\int_0^t dt_2\ e^{\Gamma_t+\Gamma_{t_2}}A_t\delta(t-t_2)\left[\tilde{x}_t,\left[\tilde{x}_{t_2},\hat{\rho}_0\right]\right]\\ &+\frac{2^2}{(i\hbar)^4}\int_0^t dt_2\int_0^{t_2}dt_3\int_0^{t_3}dt_4e^{\Gamma_t+\Gamma_{t_2}+\Gamma_{t_3}+\Gamma_{t_4}}\\ &\times\left[\tilde{x}_t,\left[\tilde{x}_{t_2},\left[\tilde{x}_{t_3},\left[\tilde{x}_{t_4},\hat{\rho}_0\right]\right]\right]\right]\\ &\times\{A_t\delta(t-t_2)A_{t_3}\delta(t_3-t_4)\\ &+A_t\delta(t-t_3)A_{t_2}\delta(t_2-t_4)\\ &+A_{t_2}\delta(t_2-t_3)A_t\delta(t-t_4)\}+\cdots.\end{aligned}$$

Due to the time ordering $t\geq t_2\geq t_3\geq t_4$, only the products of delta-functions $\delta(t-t_2)\,\delta(t_3-t_4)$ contribute to the integral, and in general,

$$\overline{F_{t_1}F_{t_2}\ldots F_{t_{2n}}}\equiv 2^n\prod_{i=1}^{n}A_{t_{2i-1}}\delta(t_{2i-1}-t_{2i}).\tag{A5}$$

Thus the terms proportional to the coefficient A_t can be combined as follows,

$$\begin{aligned}\frac{d\tilde{\rho}_t}{dt} &= \frac{A_t}{(i\hbar)^2}e^{2\Gamma_t}\left[\tilde{x}_t,\left[\tilde{x}_t,\hat{\rho}_0\right]\right]+\frac{2^2A_t}{(i\hbar)^4}\left(\frac{1}{2}\right)^2e^{2\Gamma_t}\\ &\times\int_0^t dt_3A_{t_3}e^{2\Gamma_{t_3}}\left[\tilde{x}_t,\left[\tilde{x}_t,\left[\tilde{x}_{t_3},\left[\tilde{x}_{t_3},\hat{\rho}_0\right]\right]\right]\right]+\cdots\\ &= \frac{A_t}{(i\hbar)^2}e^{2\Gamma_t}\left[\tilde{x}_t,\left[\tilde{x}_t,\tilde{\rho}_t\right]\right],\end{aligned}\tag{A6}$$

if one notices that

$$\tilde{\rho}_t=\hat{\rho}_0+\frac{1}{(i\hbar)^2}\int_0^t dt_3\ A_{t_3}e^{2\Gamma_{t_3}}\left[\tilde{x}_{t_3},\left[\tilde{x}_{t_3},\hat{\rho}_0\right]\right]+\cdots.$$

The factor 1/2 in the first line of (A6) occurs due to the integration of the Dirac delta function over a *semi-infinite* interval. We emphasize that Eq. (A6) is *exact* under the assumption (35), because no terms were disregarded.

Returning to the original operator $\hat{\rho}_t$ with the aid of the transformation (29), we arrive at Eq. (41).

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